# HEAT EXCHANGE AND DRAG IN A TURBULENT BOUNDARY LAYER WITH PRESSURE GRADIENT

A. I. Leont'ev and V. M. Fomichev UDC 532.526.4

On the basis of a semiempirical Prandtl model of turbulence, the heat transfer and drag coefficients are determined in a turbulent boundary layer with longitudinal pressure gradient.

Investigation of gradient turbulent flows is of considerable scientific and practical interest, in connection, particularly, with the problem of heat-transfer intensification. The hydrodynamics of flows with a positive pressure gradient have been investigated most completely [1]. Flow with a negative pressure gradient, and especially heat-transfer questions, have been studied less. Up to now there has been no convincing explanation of the so-called "relaminarization" effect, which consists of a reduction in the heat transfer coefficient to values inherent in a laminar flow [2-5]. The number of theoretical papers on heat transfer in gradient flows is extremely small. They are devoted mainly to the determination of a coefficient of the Reynolds analogy [6, 7]. The dependences obtained that relate the heat-transfer coefficient to the drag coefficient are not closed, since they include the integral boundary layer characteristics. To obtain them, the velocity (temperature) profiles in the boundary layer must be known, which is a no less complex problem.

The possibilities of obtaining closed dependences that would permit computation of the heat transfer and drag if only the magnitude of the pressure gradient were known have not been investigated up to now, even for the case of the simplest turbulent shear flow model, a model based on the hypothesis of the Prandtl mixing path.

The present paper is an attempt to eliminate this gap.

Let us examine a two-layer model of a turbulent shear flow. For the turbulent core

$$\tau = \mu_T \frac{\partial U}{\partial y}, \tag{1}$$

where

$$\mu_T = \rho l^2 \frac{\partial U}{\partial y}.$$

Starting from these relationships, we can obtain an expression for the drag coefficient. Let us write it in the form

$$\Psi = \frac{2}{c_{f_0}} \left( \frac{1-\omega_1}{\int\limits_{\xi_1}^{1} \frac{\delta}{l} \tau^{1/2} d\xi} \right)^2. \tag{2}$$

Assuming $l/\delta = \varkappa \xi \tau_0^{1/2}$ [1], we convert the last relationship to

$$\Psi = \frac{2}{c_{f_0}} \left( \varkappa \frac{1-\omega_1}{I_1} \right)^2,$$

where

$$I_1 = \int\limits_{\xi_1}^{1} \frac{t}{\xi} d\xi,\ t = \left( \frac{\tilde{\tau}}{\tilde{\tau}_0} \right)^{1/2}.$$

N. E. Bauman Moscow Higher Technical School. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 45, No. 1, pp. 5-11, July, 1983. Original article submitted March 16, 1982.

  

Here $\xi_1 = y_1/\delta$ and $\omega_1 = U_1/U_\infty$ are dimensionless values of the viscous sublayer thickness and of the velocity on its boundary, $\Psi = c_f/c_{f0}$ is the relative drag coefficient, and $\tau_0$, $c_{f0}$ are the tangential stress and the drag coefficient of the turbulent boundary layer under "standard conditions," i.e., on a flat semiinfinite plate, $\tilde{\tau} = \tau/\tau_W$.

The expression for the heat flux in the turbulent core is

$$q = -c_p \frac{\mu_T}{\mathrm{Pr}_T} \frac{\partial T}{\partial y} . \tag{3}$$

A result of this equality and of (1) is the relationship

$$\frac{c_f}{2\mathrm{St}} \frac{d\theta}{d\xi} = -\mathrm{Pr}_T \frac{d\omega}{d\xi} \frac{\tilde{q}}{\tilde{\tau}} .$$

Here $\theta = (T - T_W)/(T_\infty - T_W)$, $\tilde{q} = q/q_W$ is the distribution of the dimensionless heat flux in the turbulent core. It is known that this distribution possesses the conservativity property relative to the pressure gradient. Hence, without substantial error, it can be assumed that $\tilde{q} = \tilde{q}_0$. Also assuming that $\tilde{q}_0 = \tilde{\tau}_0$ is valid for $dP/dx = 0$, we integrate (3) between the limits $\xi_1$ and 1. We obtain

$$\frac{\Psi_T}{\Psi} = \frac{1 - \theta_1}{\int\limits_0^1 \frac{\tilde{\tau}_0}{\tilde{\tau}} \frac{d\omega}{d\xi} d\xi} , \quad r = \mathrm{Pr}_T \Big/ \int\limits_{\xi_1}^1 \mathrm{Pr}_T d\xi.$$

Here $\Psi = \mathrm{St}/\mathrm{St}_0$ is the relative heat-transfer coefficient, and $\theta_1$ is the dimensionless temperature on the viscous sublayer boundary.

Taking into account that

$$\frac{d\omega}{d\xi} = \Psi^{1/2} \left(\frac{c_{f_0}}{2}\right)^{1/2} \frac{t}{\varkappa\xi} ,$$

we find

$$\frac{\Psi_T}{\Psi^{1/2}} = \varkappa \left(\frac{2}{c_{f_0}}\right)^{1/2} \frac{1 - \theta_1}{I_2} , \tag{4}$$

where

$$I_2 = \int\limits_{\xi_1}^1 \frac{r}{t\xi} d\xi.$$

Starting from (4), the following comparative relationships can be obtained between the heat transfer and the drag coefficients. If $t = 1$ and $r = 1$, we obtain $\Psi_T = \Psi = 1$. For diffusion flows ($dP/dx > 0$) $t > 1$, and therefore, $\Psi_T > \Psi^{1/2}$. For contracting flows ($dP/dx < 0$) $t < 1$, and therefore $\Psi_T < \Psi^{1/2}$.

A result of the relationships (2) and (4) is an expression for the relative heat-transfer law

$$\Psi_T = \frac{2\varkappa^2}{c_{f_0}} \frac{(1 - \theta_1)(1 - \omega_1)}{I_1 I_2} . \tag{5}$$

Therefore, the determination of the relative drag and heat-transfer coefficients is associated with the evaluation of the two integrals $I_1$ and $I_2$. This requires giving the function $t$ a specific form.

Flow with Positive Pressure Gradient. For this flow the tangential stress distribution over the boundary layer thickness is taken approximated by the polynomial

$$\tilde{\tau} = (1 - \xi)^2 (1 + 2\xi + \Lambda\xi), \quad \Lambda = -\frac{2}{c_f} \frac{\delta}{U_\infty} \frac{dU_\infty}{dx} , \tag{6}$$

and therefore

$$t = 1 + \frac{\Lambda\xi}{1 + 2\xi}.$$

For such a form of the function t, the integrals $I_1$ and $I_2$ are evaluated in quadratures for r = 1:

$$I_1 = \sigma + F_1(\Lambda, \xi_1), \quad I_2 = \sigma + F_2(\Lambda, \xi_1),$$

$$F_1 = 2\ln\left[3^{\frac{1-\lambda}{2}} \frac{1+\lambda_2}{2}\left(\frac{1+\lambda}{\lambda_2+\lambda}\right)^{\lambda}\right],$$

$$F_2 = 2\ln\left[3^{\frac{\lambda-1}{2\lambda}} \frac{1+\lambda_2}{2}\left(\frac{1+\lambda}{\lambda_2+\lambda}\right)^{1/\lambda}\right].$$

Here

$$\lambda = \left(1 + \frac{\Lambda}{2}\right)^{1/2}, \quad \lambda_1 = (1 + \Lambda\xi_1)^{1/2},$$

$$\lambda_2 = \left(1 + \frac{\Lambda}{3}\right)^{1/2}, \quad \sigma = -\ln\xi_1.$$

If it is assumed that

$$1 - \omega_1 = 1 - \theta_1 = \frac{\sigma}{\varkappa}\left(\frac{c_{f_0}}{2}\right)^{1/2}$$

holds approximately, we obtain the following formulas to evaluate the drag and heat transfer:

$$\Psi^{-1/2} = 1 + \frac{F_1(\Lambda, \xi_1) - 2\ln\left(\frac{1+\lambda_1}{2}\right)}{\sigma}, \tag{7}$$

$$\Psi_T^{-1} = \Psi^{-1/2}\left(1 + \frac{F_2(\Lambda, \xi_1) + 2\ln\left(\frac{1+\lambda_1}{2}\right)}{\sigma}\right). \tag{8}$$

If the state of the boundary layer is not too close to preseparation, i.e., $\Lambda\xi_1 \lesssim 1$, then (7) and (8) are simplified:

$$\Psi = \frac{1}{(1 + F_1/\sigma)^2}, \tag{9}$$

$$\Psi_T = \frac{\Psi^{1/2}}{1 + F_2/\sigma}. \tag{10}$$

The dependences $F_1$ and $F_2$ are presented in Fig. 1a.

Flow with Negative Pressure Gradient. For contracting flows the polynomial approximation of the tangential stress (6) becomes unacceptable. In fact, it is not difficult to establish that for $\Lambda < -3$ the function $\tilde{\tau}$ is negative, which has no meaning physically.

Following [6], we take

$$\tilde{\tau} = \exp(\Lambda\xi(1 - \xi) + \xi^2 \ln\varepsilon),$$

where $\varepsilon$ is a small indefinite number such that for $\xi = 1$ we have $\tilde{\tau} = \varepsilon$. Then

$$t = \exp[-\alpha\xi(1 - \xi)], \quad \alpha = -\Lambda/2.$$

Quadratures for $I_1$ and $I_2$ do not exist for such a function t. We write

$$I_1 = \sigma\exp(-\alpha\xi_1) - \tilde{F}_1(\Lambda, \xi_1), \quad I_2 = \sigma\exp(\alpha\xi_1) - \tilde{F}_2(\Lambda, \xi_1),$$

where

$$\tilde{F}_1(\Lambda, \xi_1) = -2\alpha\exp(-\alpha/4)\int_{-1/2+\xi_1}^{1/2} \exp(\alpha z^2)\, z\ln(z + 1/2)\, dz,$$

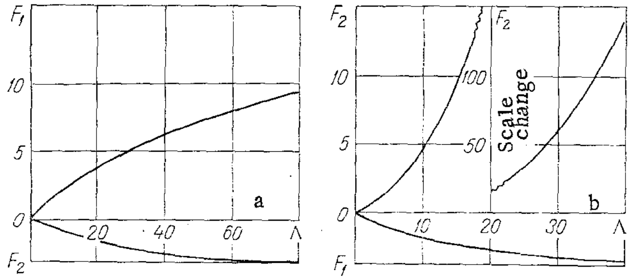


Fig. 1. The dependences $F_1(\Lambda)$ and $F_2(\Lambda)$: a) positive pressure gradient; b) negative pressure gradient.

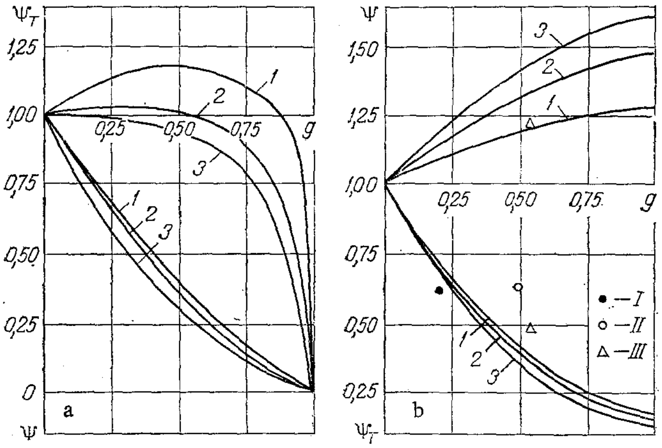


Fig. 2. Dependences of the relative drag $\Psi$ and heat transfer $\Psi_T$ coefficients on the dimensionless pressure gradient g for different values of Re**: 1) Re** = $10^3$; 2) $10^4$; 3) $10^5$; a) positive pressure gradient; b) negative; I, II, III are experimental data from [2], [3], [5], respectively.

$$\tilde{F}_2(\Lambda,\ \xi_1) = 2\alpha \exp(\alpha/4) \int\limits_{-1/2+\xi_1}^{1/2} z \exp(-\alpha z^2) \ln(z + 1/2)\, dz.$$

It can be shown that

$$\tilde{F}_2(\Lambda,\ \xi_1) = F_2(\Lambda)(1 + O(\xi_1)),$$

where $F_2(\Lambda) = \tilde{F}_2(\Lambda,\ \xi_1 = 0)$.

For large values of $\Lambda$ the more correct estimate

$$\tilde{F}_1(\Lambda,\ \xi_1) = F_1(\Lambda)(1 + O(\xi_1)) - \frac{\alpha\xi_1}{2}(1 + \sigma)$$

should be used for the function $\tilde{F}_1$. Formulas to calculate the drag and heat transfer take the form

$$\Psi^{-1/2} = \exp(-\alpha\xi_1) + \frac{\alpha\xi_1}{2}\left(1 + \frac{1}{\sigma}\right) + \frac{F_1}{\sigma}, \tag{11}$$

$$\Psi_T^{-1} = \Psi^{-1/2}\left[\exp(\alpha\xi_1) - \frac{\alpha\xi_1}{2}\left(1 + \frac{1}{\sigma}\right) + \frac{F_2}{\sigma}\right]. \tag{12}$$

The dependences $F_1(\Lambda)$ and $F_2(\Lambda)$ are presented in Fig. 1b.

If $\Lambda\xi_1 < 1$, which is valid even for sufficiently high stream accelerations, (11) and (12)

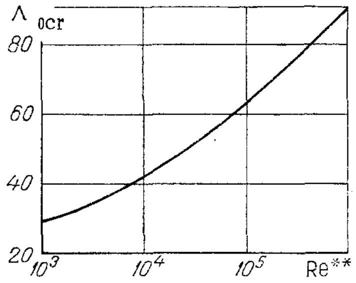


Fig. 3. Dependence of the boundary layer separation parameter $\Lambda_{0cr}$ on the number Re**.

take on the universal form (9) and (10). For large values of the parameter $\alpha \geqslant 10$ the integral $F_2$ is evaluated approximately as

$$F_2 = -\exp(\alpha/4) \sum_{m=1}^{\infty} \frac{\Gamma(m+1/2)}{\alpha^{m-1/2}} \frac{4^m}{2m-1},$$

where $\Gamma(m + 1/2)$ is the gamma function. Limiting ourselves to the first four terms of the series, we obtain

$$F_2 = 4\Gamma(3/2)\exp(\alpha/4)\frac{\alpha+2}{\alpha^{3/2}}, \quad \Gamma(3/2) = 0.886.$$

Results of the Computations. In order to compute the drag and heat transfer by means of (7)-(12), there remains to determine the dimensionless viscous sublayer thickness $\xi_1$. From the equality

$$\dot{\mathrm{Re}} = \left(\frac{\rho y^2}{\mu} \frac{\partial U}{\partial y}\right)_{y=y_1}$$

there follows that in the absence of a pressure gradient

$$\xi_{10} = \frac{\tilde{\delta}^{**}}{\mathrm{Re}^{**}}\left(\dot{\mathrm{Re}}\frac{2}{c_{f_0}}\right)^{1/2},$$

where $\dot{\mathrm{Re}}$ is the so-called stability criterion for the viscous sublayer. Assuming [1]

$$\dot{\mathrm{Re}} = 134, \quad \frac{c_{f_0}}{2} = 0.0128\mathrm{Re}^{**^{-0.25}},$$

we obtain

$$\xi_{10} = 102\tilde{\delta}^{**}\mathrm{Re}^{**^{-0.875}}, \quad \ln\xi_{10} = 2.3 - 2.01\lg\mathrm{Re}^{**}.$$

Results of computing the viscous sublayer thickness in the presence of a positive pressure gradient are presented in [1]. It follows from there that even for boundary layer separation, when the difference between $\xi_1$ and $\xi_{10}$ is maximal,

$$(\ln\xi_1 - \ln\xi_{10})/\ln\xi_1 < 0.25,$$

in a broad range of numbers $\mathrm{Re}^{**} \leqslant 10^6$. Hence, it can approximately be assumed $\varkappa = -\ln\xi_{10}$.

Dependences of the relative drag and heat-transfer coefficients on the parameter $g = \Lambda_0/\Lambda_{0cr}$ are shown in Fig. 2 for $dP/dx > 0$, and on parameter $g = -\Lambda/\Lambda_{0cr}$ for $dP/dx < 0$, for different numbers $\mathrm{Re}^{**} = 10^3$-$10^5$. Here $\Lambda_0 = \Psi\Lambda = 2/c_{f_0}\, \delta/U_\infty\, dU_\infty/dx$, while $\Lambda_{0cr}$ is the critical value at which $c_f = 0$. The dependence is presented in Fig. 3.

We first examine the results of diffusor flows. The dependences of the relative drag coefficient on the parameter are in satisfactory agreement with the analogous dependences obtained earlier on a digital computer, which can be approximated by the relationship [1]

$$\Psi = (1-g)^{1.54}.$$

The heat-transfer data have been obtained for the first time. It is seen that if the boundary layer is not close to separation, the influence of the positive pressure gradient on the heat transfer is negligible. For moderate values of the number $Re^{**} \sim 10^3$, even its small increase holds.

As follows from the results of the computations presented in Fig. 2b, for contracted flows, on the other hand, an increase in the drag coefficient and a diminution in the heat-transfer coefficient is observed. It is characteristic that the dependence $\Psi(g)$ is practically invariant in a broad range of the numbers $Re^{**} = 10^3$-$10^5$. It can be described approximately by the relationship

$$\Psi_T = 1 - 0.93g^{2/3}.$$

Test data on the drag and heat transfer borrowed from [2]-[4] are also presented in Fig. 2b. It should be noted that they are all obtained for comparatively small numbers $Re^{**} \sim 10^3$. Satisfactory correspondence is observed between the theoretical computations and the experimental data.

Therefore, it is shown that the so-called "relaminarization" effect for a turbulent boundary layer during stream acceleration can be explained on the basis of the classical semi-empirical Prandtl theory of turbulence.

## NOTATION

x,y, longitudinal and transverse coordinates; $\tau$, tangential stress; q, heat flux; U, stream velocity; T, temperature; $\mu$, viscosity; $\rho$, density; $c_f$, drag coefficient; St, Stanton criterion; Pr, Prandtl number; $Re^{**}$, Reynolds number calculated with respect to the thickness of the loss of momentum; $\Lambda, \Lambda_0$, pressure gradient parameters. The subscripts are: $\infty$, external flow; w, wall; 1, for viscous sublayer; and 0, for standard conditions.

## LITERATURE CITED


1. S. S. Kutateladze and A. I. Leont'ev, Heat and Mass Transfer and Friction in a Turbulent Boundary Layer [in Russian], Énergiya, Moscow (1972).
2. W. M. Kays et al., "Heat transfer to highly accelerated turbulent boundary layer with and without mass addition," Trans. ASME, Ser. C, 92, No. 3, 190-198 (1970).
3. C. H. Back et al., "Laminarization of a turbulent boundary layer in nozzle flow. Boundary layer and heat-transfer measurement with wall cooling," Trans. ASME, Ser. C, 92, No. 3, 29-40 (1970).
4. A. B. Zhalyauskas and A. A. Pyadyushis, "Momentum and heat transfer in a turbulent boundary layer under stream acceleration," Problems of Turbulent Transfer [in Russian], Inst. Heat and Mass Transfer Acad. Sci. BSSR, Minsk (1979), pp. 123-133.
5. E. U. Repik and V. K. Kuzenkov, "Experimental investigation of the relation between heat transfer and friction drag in a turbulent boundary layer with longitudinal pressure gradient," Teplofiz. Vys. Temp., No. 6, 1197-1202 (1980).
6. N. Tetervin, "Approximate calculation of Reynolds analogy for turbulent boundary layer with pressure gradient," AIAA J., No. 6, 1079-1085 (1969).
7. V. K. Migai, "Reynolds analogy in a turbulent boundary layer with pressure gradient," Heat and Mass Transfer-V [in Russian], Vol. 1, Pt. 1 (1976), pp. 148-151.